JOURNAL OF APPROXIMATION THEORY 29, 103–115 (1980)

Comparison between the Degrees of Approximation by Lacunary and Ordinary Algebraic Polynomials

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Communicated by G. G. Lorentz

Received June 1, 1978

DEDICATED TO THE MEMORY OF P. TURÁN

I. INTRODUCTION

Let $C[a, b]$ denote the space of real valued continuous functions defined on $[a, b]$, endowed with the uniform norm denoted by $\| \|$. Let $E_n(f)$ be the distance between the function $f \in C[a, b]$ and the subspace of algebraic polynomials of degree at most n and let $E_n^k(f)$ be the distance from f to the subspace of algebraic polynomials of degree at most n in which the coefficient of x^k is 0. This paper is devoted to the following problem: find the functions $f \in C[a, b]$ for which

$$\lim_{n\to\infty} \frac{E_n^k(f)}{E_n(f)} = \infty, \qquad k \geqslant 1. \tag{1.1}$$

Our work originated in a paper of Bak and Newman [1] on Muntz's theorem. This theorem [9, p. 197] states that the polynomials of the form $\sum_{k=0}^n a_k x^{\lambda_k}$ are dense in $C[0, 1]$ if

$$0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

In [1] the degree of convergence of such polynomials to a function $f \in C[0, 1]$ is related to the modulus of continuity of f, $\omega(f, \cdot)$. Let $\lambda_k - \lambda_{k-1} \geqslant 2$ in the above sequence and let $E_n^{\Lambda}(f) = d(f, [1, x^{\lambda_1}, x^{\lambda_2},..., x^{\lambda_n}])$ be the distance from f to the space generated by $(1, x^{\lambda_1}, x^{\lambda_2},..., x^{\lambda_n})$. Then

$$E_n^{\Lambda}(f) \leqslant M\omega\left(f, \exp\left(-2 \sum_{k=1}^{n} \lambda_k\right)\right)$$

for some constant M which does not depend on f. If $f(x) = | x - \tfrac{1}{2} |$,

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0021-9045/80/060103–13$02.00/0

640/29/2-2

$x \in [0, 1]$, this theorem implies that $E_n^k(f) \leqslant K/n$ for a constant K independent of n. Indeed, let k be odd. Then

$$E_n^k(f) \leqslant d(f, [1, x^2, x^4, ..., x^{2[n/2]}]) \leqslant M\omega\left(f, \exp\left(-2 \sum_{k=1}^{[n/2]} \frac{1}{2k}\right)\right)$$
$$\leqslant N\omega\left(f, \frac{1}{n}\right) \leqslant \frac{K}{n}.$$

The proof is similar when k is even. Also we know [5, p. 171] that $E_n(f) \geqslant N/n$ with a constant N which does not depend on n. We conclude that, for every integer $k \geqslant 1$,

$$\overline{\lim_{n\to\infty}} \frac{E_n^k(|x - \frac{1}{2}|)}{E_n(|x - \frac{1}{2}|)} < \infty. \tag{1.2}$$

On the other hand, the classical proof of Muntz's theorem is based on the formula [9, p. 196] which gives the distance d_n, in $L_2[0, 1]$, between x^n and $x^{p_1}, x^{p_2}, ..., x^{p_n}]$, where $p_i > -\frac{1}{2} \; \forall \, i$:

$$d_n = \frac{1}{(2m + 1)^{1/2}} \prod_{j=1}^{n} \frac{|m - p_j|}{m + p_j + 1}.$$

Now, let P_n, Q_n be polynomials of degree at most n such that

$$\|x - P_n(x)\| = E_n^1(x)$$

and

$$\|x - Q_n(x)\|_{L_2[0,1]} = d_{L_2}(x, [1, x, ..., x^n]).$$

We have

$$E_n^1(x) = \|x - P_n(x)\| \geqslant \|x - P_n(x)\|_{L_2} \geqslant \|x - Q_n(x)\|_{L_2}$$
$$= \frac{1}{3^{1/2}} \frac{1}{4} \frac{2}{5} \frac{3}{6} \cdots \frac{n-1}{n+2} \geqslant \frac{K}{n^3}, \qquad n > 1,$$

for some constant K independent of n. Clearly $E_n(x) = 0$, $n \geqslant 1$.

These were the observations which led us to conjecture that, given k, (1, 1) holds if $f \in C^r[0, 1]$ for r large enough, where $C^r[0, 1]$ is the subspace of $C[0, 1]$ of r-times continuously differentiable functions. Indeed, (1, 2) shows that f must be sufficiently smooth in order for (1, 1) to hold.

The following notations will be used throughout: If $f \in C[a, b]$ and if $a < a' < b' < b$, $E_n(f, [a', b'])$ denotes the degree of uniform approximation of $f|_{[a',b']}$ by polynomials of degree at most n. We write $\|f\|_{[a',b']}$ for $\sup_{x\in[a',b']} |f(x)|$. Also P_n, Q_n will always stand for algebraic polynomials of degree at most n.

II. The Problem of Computing $Ee(x_a)$

One of the basic tools for the investigation of the asymptotic behaviour of $E_n^k(f)/E_n(f)$ is knowledge of $E_n^k(x^k)$.

Theorem 2.1. *Let k be an integer $\geqslant 1$. Then there exist positive constants N_k and M_k with the following property: for every integer $n \geqslant 1$,*

$$\frac{N_k}{n^{2k}} \leqslant E_n^k(x^k, [0, 1]) \leqslant \frac{M_k}{n^{2k}}.$$

The proof relies on the following lemmas.

Lemma 2.2. $\{1, x, \ldots, x^{k-1}, x^{k+1}, \ldots, x^n\}$, $1 \leqslant k < n$, *is a Chebychev system on* $[0, 1]$.

Proof. It follows from Rolle's theorem.

Lemma 2.3. $E_n^k(x^k, [0, 1]) = k!/|2^k T_n^{(k)}(-1)|$, $1 \leqslant k < n$, *where* $T_n(x) = \cos(n \arccos x)$ *is the nth Chebychev polynomial.*

Proof. There exist $n + 1$ points on $[-1, 1]$ where T_n takes the values $\pm\| T_n \|_{[-1,1]} = \pm 1$ with alternating signs. So there exist $n + 1$ points on $[0, 1]$ where $P_n(x) = T_n(2x - 1)$ takes the values $\pm\| P_n \|_{[0,1]} = \pm 1$ with alternating signs. It follows from Chebychev's alternation theorem [4, p. 30] and the preceding lemma that $E_n^k(x^k, [0, 1]) = \| -(1/a_k) P_n(x) + x^k - x^k \|_{[0,1]} = 1/| a_k |$, where a_k is the coefficient of x^k in P_n, and the lemma follows.

Lemma 2.4.

$$| T_n^{(k)}(-1)| = \prod_{i=1}^{k} \frac{n^2 - (i-1)^2}{(2k-1)!!},$$

where $(2k - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2k - 1)$.

Proof. $| T_n^{(k)}(-1)| = | T_n^{(k)}(1)|$ because T_n is either odd or even, and $T_n^{(k)}(1)$ equals the above product [7, p. 226].

Theorem 2.1 follows now from Lemmas 2.3 and 2.4.

Theorem 2.5. *Let k be an integer $\geqslant 1$. There exist positive constants N_k and M_k such that, for every integer $n \geqslant 1$,*

$$\frac{N_k}{n^k} \leqslant E_n^k(x^k, [-1, 1]) \leqslant \frac{M_k}{n^k}.$$

Proof. We first show that $E_n{}^k(x^k, [-1, 1]) \leqslant M_k/n^k$. Suppose that $n \equiv k \pmod 2$. Let $P_n(x) = (k!/T_n^{(k)}(0))\, T_n(x)$. Then

$$E_n{}^k(x^k, [-1, 1]) \leqslant \| -P_n(x) + x - x \| = \frac{k!}{|T_n^{(k)}(0)|}.$$

Now, from the relation [7, p. 226, Eq. (47)]

$$-T_m^{(k+1)}(0) = (m^2 - (k-1))\, T_m^{(k-1)}(0),$$

and from

$$T_{2m}(0) = (-1)^m, \qquad T'_{2m+1}(0) \doteq (-1)^n\, m, \qquad m \geqslant 0,$$

we find that

$$|T_n^{(k)}(0)| \geqslant K_k n^k, \qquad n \geqslant 1.$$

It follows that

$$E_n{}^k(x^k, [-1, 1]) \leqslant \frac{K_k'}{n^k}, \qquad n \geqslant 1.$$

Suppose now that $n \not\equiv k \pmod 2$. We have:

$$E_n{}^k(x^k, [-1, 1]) \leqslant E_{n-1}^k(x^k, [-1, 1]) \leqslant \frac{K_k'}{(n-1)^k}$$
$$\leqslant \frac{K_k''}{n^k}, \qquad n \geqslant 2.$$

It follows that

$$E_n{}^k(x^k, [-1, 1]) \leqslant \frac{M_k}{n^k}, \qquad n = 1, 2, \ldots . \tag{2.1}$$

We now show the existence of a constant N_1 such that

$$E_n{}^1(x, [-1, 1]) \geqslant \frac{N_1}{n}, \qquad n = 1, 2, \ldots .$$

Let P_n, $P_n'(0) = 0$, satisfy

$$\| P_n(x) - x \| = E_n{}^1(x) \leqslant \frac{M_1}{n}, \qquad n \geqslant 1.$$

Now

$$\| P_n'(x) - 1 \|_{[-1/2, 1/2]} \leqslant K_1 n \| P_n(x) - x \|_{[-1,1]} \leqslant K_1 M_1$$

by Bernstein's inequality. It follows that

$$\| P_n'(x)\|_{[-1/2,1/2]} \leqslant M, \qquad n = 1, 2,\ldots,$$

and

$$\| P_n''(x)\|_{[-1/4,1/4]} \leqslant Kn, \qquad n = 1, 2,\ldots,$$

again by Bernstein's inequality. So we have

$$P_n'(x) \leqslant \frac{1}{2}, \quad x \in \left[0, \frac{1}{2Kn}\right], \qquad n \geqslant \frac{2}{K},$$

by the mean value theorem and the fact that $P_n'(0) = 0$.

Again, by the mean value theorem,

$$P_n(x) - P_n(0) \leqslant \frac{x}{2}, \quad x \in \left[0, \frac{1}{2Kn}\right], \qquad n \geqslant \frac{2}{K}. \tag{2.2}$$

Suppose that

$$P_n(0) \leqslant \frac{1}{8Kn}.$$

Then (2.2) implies

$$\left| P_n\left(\frac{1}{2Kn}\right) - \frac{1}{2Kn} \right| \geqslant \frac{1}{8Kn}.$$

We have proved that

$$\| P_n(x) - x \|_{[0,1/2Kn]} \geqslant \frac{1}{8Kn},$$

so that

$$E_n{}^1(x, [-1, 1]) \geqslant \frac{N_1}{n}.$$

We remark that we have actually proved: let P_n be a sequence of polynomials with $P_n'(0) = 0$ and $\| P_n(x) - x \|_{[-a,a]} \leqslant C/n$. Then $\| P_n(x) - x \|_{[-a,a]} \geqslant D/n$ $(0 < a \leqslant 1)$. Now, let k be an integer $\geqslant 2$ and let P_n be a polynomial with $P_n^{(k)}(0) = 0$ and

$$\| P_n(x) - x^k \| = E_n{}^k(x^k, [-1, 1]). \tag{2.4}$$

We have, by repeatedly applying Bernstein's inequality,

$$\begin{aligned} &\| P_n(x) - x^k \|_{[-1,1]} \\ &\quad \geqslant \frac{K_1}{n} \| P_n'(x) - kx^{k-1} \|_{[-1+1/k, 1-1/k]} \geqslant \cdots \\ &\quad \geqslant \frac{K_1 K_2 \cdots K_{k-1}}{n(n-1)\cdots(n-(k-2))} \| P_n^{(k-1)}(x) - k!\, x \|_{[-1+(k-1)/k, 1-(k-1)/k]}. \end{aligned} \tag{2.5}$$

But, again by Bernstein's inequality and (2.1), we have

$$\| P_n^{(k-1)}(x) - k!\, x \|_{[-1+(k-1)/k, 1-(k-1)/k]} \leqslant \frac{C_k}{n}.$$

The above remark and the fact that $P_n^{(k)}(0) = 0$ yield

$$\| P_n^{(k-1)}(x) - k!\, x \|_{[-1+(k-1)/k, 1-(k-1)/k]} \geqslant \frac{D_k}{n}. \tag{2.6}$$

(2.4), (2.5) and (2.6) show the existence of a constant N_k such that, for every integer $n \geqslant 1$ and for $k \geqslant 2$,

$$E_n{}^k(x^k, [-1, 1]) \geqslant \frac{N_k}{n^k}. \tag{2.7}$$

By (2.3), (2.7) is also true for $k = 1$. The proof of Theorem 2.5 is complete.

THEOREM 2.6. *Let $a < b$ and either $a = 0$ or $b = 0$. Let k be an integer $\geqslant 1$. Then there exist constants M_k, N_k such that*

$$\frac{N_k}{n^{2k}} \leqslant E_n{}^k(x^k, [a, b]) \leqslant \frac{M_k}{n^{2k}}, \qquad n = 1, 2,\ldots.$$

Proof. Suppose $a = 0$. The polynomial $P_n(x) = T_n(2x/(b-a) - 1)$ has the alternation property (cf. proof of Lemma 2.3) on $[a, b]$ and $\{1, x,\ldots, x^{k-1}, x^{k+1},\ldots, x^n\}$ is a Chebychev system on $[a, b]$. The proof of Lemma 2.3 shows that $E_n{}^k(x^k, [a, b]) = k!/|P_n^{(k)}(0)|$ and $P_n^{(k)}(0) = (2^k/(b-a)^k)\, T_n^{(k)}(-1)$. The theorem follows by Lemma 2.4. The proof is similar if $b = 0$.

THEOREM 2.7. *Let $a < 0 < b$. Let k be an integer $\geqslant 1$. Then there exist constants M_k, N_k such that*

$$\frac{N_k}{n^k} \leqslant E_n{}^k(x^k, [a, b]) \leqslant \frac{M_k}{n^k}, \qquad n = 1, 2,\ldots.$$

Proof. The proof of Theorem 2.5 shows that our assertion holds for an interval $[-\alpha, \alpha]$ ($\alpha > 0$). The theorem follows from the relation

$$E_n^k(x^k, [-\alpha, \alpha]) \leqslant E_n^k(x^k, [a, b]) \leqslant E_n^k(x^k, [-\beta, \beta])$$

where $\alpha = \min(|a|, |b|)$ and $\beta = \max(|a|, |b|)$.

Remark. If $0 \notin [a, b]$, then $E_n^k(x^k, [a, b]) \to 0$ as $n \to \infty$, at an exponential rate. Indeed $\{1, x,..., x^{k-1}, x^{k+1},..., x^n\}$ is a Chebychev system on $[a, b]$. Consider the polynomial $P_n(x) = T_n(2(x - a)/(b - a) - 1)$. If $a > 0$ or $b < 0$ (and $a < b$), then $-1 < -1 - 2a/(b - a)$, and our claim reduces to estimating $T^{(k)}$ at that point. From the fact that $T_n(x) = \cosh(n \operatorname{arc} \cosh x)$ for $x > 1$ [6, p. 5], we see that $T_n^{(k)}(\alpha)$ grows exponentially for $|\alpha| > 1$. The assertion follows.

Let us notice that a good asymptotic majorant of $E_n^k(x^k, [0, 1])$ could have been derived from a proof of Muntz's theorem [8], or by using methods of functional analysis [2, p. 125]. However, these techniques do not yield a good minorant which will be needed. Moreover, these techniques do not seem to yield any information on $E_n^k(x^k, [-1, 1])$.

III. Asymptotic Behavior of $E_n^k(f)/E_n(f)$

The theorems of Section II and knowledge of the behavior of the derivatives of polynomials of best approximation [3] will be our tools in the investigation of this problem.

The purpose of this article is proving the following four theorems. (In this section, f and g are not polynomials.)

Theorem 3.1. *Let k be an integer $\geqslant 1$ and let $f \in C^{2k}[a, b]$, where $a = 0$ or $b = 0$, and $f^{(k)}(0) \neq 0$. Then*

$$\lim_{n\to\infty} \frac{E_n^k(f)}{E_n(f)} = \infty.$$

More precisely, there exists a constant M which depends only on a, b and k such that, for every integer $n > 2k$,

$$\frac{E_n^k(f)}{E_n(f)} \geqslant \frac{M}{E_{n-2k}(f^{(2k)})}.$$

This theorem cannot be improved in the sense that:

THEOREM 3.2. *For every integer $N \geqslant 0$ there exists a function $g \in C^N[a, b]$, $a = 0$ or $b = 0$, such that*

$$\overline{\lim_{n\to\infty}} \frac{E_n^k(g)}{E_n(g)} < \infty, \qquad k \geqslant \left[\frac{N}{2}\right] + 1.$$

THEOREM 3.3. *Let k be an integer $\geqslant 1$ and let $f \in C^k[a, b]$, where $a < 0 < b$ and $f^{(k)}(0) \neq 0$. Then*

$$\lim_{n\to\infty} \frac{E_n^k(f)}{E_n(f)} = \infty.$$

More precisely, there exists a constant M which depends only on a, b and k such that, for every integer $n > k$,

$$\frac{E_n^k(f)}{E_n(f)} \geqslant \frac{M}{E_{n-k}(f^{(k)})}.$$

This theorem cannot be improved in the sense that:

THEOREM 3.4. *For every integer $N \geqslant 0$ there exists a function $g \in C^N[a, b]$, $a < 0 < b$, such that*

$$\overline{\lim_{n\to\infty}} \frac{E_n^k(g)}{E_n(g)} < \infty, \qquad k \geqslant N + 1.$$

LEMMA 3.5. *Let $f \in C^k[a, b]$, $k \geqslant 1$, let $a_k = f^{(k)}(0)/k!$ and let a_n^k be the coefficient of x^k in the polynomial of degree at most n of best approximation to f on $[a, b]$. Then*

$$E_n^k(f) \geqslant -\mid a_n^k - a_k \mid E_n^k(x^k) - E_n(f) + \mid a_k \mid E_n^k(x^k).$$

Proof. From the definitions of $E_n(f)$, $E_n^k(f)$ and a_n^k we obtain $E_n^k(f(x) - a_n^k x^k) = E_n(f(x))$. Now

$$\begin{aligned} E_n^k(-a_k x^k) &= E_n^k(-a_k x^k + f(x) - f(x) + a_n^k x^k - a_n^k x^k) \\ &\leqslant E_n^k(f(x) - a_n^k x^k) + E_n^k((a_n^k - a_k)x^k) + E_n^k(f(x)) \\ &\leqslant E_n(f(x)) + \mid a_n^k - a_k \mid E_n^k(x^k) + E_n^k(f(x)). \end{aligned}$$

The lemma follows.

Proof of Theorem 3.1. Theorem 2.4 in [3] implies the existence of S_k, independent of n, such that

$$\mid a_n^k - a_k \mid \leqslant S_k E_{n-2k}(f^{(2k)}), \qquad n > 2k.$$

By Theorem 2.6 we know that there exists an N_k independent of n such that

$$E_n^k(x^k) \geqslant \frac{N_k}{n^{2k}}, \qquad n \geqslant 1.$$

So, by Lemma 3.5,

$$\frac{E_n^k(f)}{E_n(f)} \geqslant -1 - \frac{S_k N_k E_{n-2k}(f^{(2k)})}{n^{2k} E_n(f)} + \frac{N_k \mid a_k \mid}{n^{2k} E_n(f)}.$$

But $\lim_{n\to\infty} E_{n-2k}(f^{(2k)}) = 0$ and $a_k \neq 0$. Hence

$$\frac{E_n^k(f)}{E_n(f)} \geqslant \frac{R_k}{n^{2k} E_n(f)} - 1, \qquad n \geqslant 1, \tag{3.1}$$

for some R_k independent of n. Since $f \in C^{2k}[a, b]$, Jackson's theorem [4, p. 127] implies that

$$E_n(f) = o\left(\frac{1}{n^{2k}}\right).$$

So we have

$$\lim_{n\to\infty} \frac{E_n^k(f)}{E_n(f)} = \infty.$$

As [2, p. 39] there exists a constant K such that, for $f \in C^1[a, b]$,

$$E_n(f) \leqslant \frac{K}{n} E_{n-1}(f'),$$

the theorem follows from (3.1).

LEMMA 3.6. *Let* $f \in C^N[a, b]$, $a = 0$ or $b = 0$, $k \geqslant [N/2] + 1$. *There exists a constant* K_k *such that*

$$E_n^k(f) \leqslant K_k \frac{1}{n^N} \omega\left(f^{(N)}, \frac{1}{n}\right), \qquad n = 1, 2, \ldots .$$

Proof. Let a_n^k be as in Lemma 3.5. Then

$$E_n^k(f) \leqslant E_n(f) + \mid a_n^k \mid E_n^k(x^k).$$

Indeed, $E_n^k(f) \leqslant E_n^k(f(x) - a_n^k x^k) + E_n^k(a_n^k x^k)$ and $E_n^k(f(x) - a_n^k x^k) = E_n(f(x))$.

Theorems 3.1 and 3.2 in [3] imply that

$$| a_n^k | \leqslant M_k n^{2k-N} \omega \left(f^{(N)}, \frac{1}{n}\right).$$

Thus

$$\frac{E_n^k(f)}{n^{-N}\omega(f^{(N)}, 1/n)} \leqslant \frac{E_n(f)}{n^{-N}\omega(f^{(N)}, 1/n)} + M_k N_k \frac{n^{2k-N}\omega(f^{(N)}, 1/n)}{n^{2k}n^{-N}\omega(f^{(N)}, 1/n)}.$$

But by Jackson's theorem [2, p. 39], $E_n(f)/n^{-N}\omega(f^{(N)}, 1/n)$ is bounded. The lemma is proved.

Proof of Theorem 3.2. Let $g(x) = (x - (b-a)/2)^N \mid x - (b-a)/2 \mid$. It is known [7, p. 410] that

$$E_n(g) \geqslant \frac{K_1}{n^{N+1}}.$$

On the other hand,

$$E_n^k(g) \leqslant \frac{K_2}{n^N} \omega \left(g^{(N)}, \frac{1}{n}\right) \leqslant \frac{K_2}{n^N} \frac{K_3}{n}$$

by the preceding lemma. Theorem 3.2 follows.

Proof of Theorem 3.3. Theorem 2.8 in [3] implies the existence of S_k independent of n, such that

$$| a_n^k - a_k | \leqslant S_k E_{n-k}(f^{(k)}), \qquad n > k,$$

where a_n^k and a_k are as in Lemma 2.5. By Theorem 2.7 we know that there exists an N_k, independent of n, such that

$$E_n^k(x^k) \geqslant \frac{N_k}{n^k}, \qquad n \geqslant 1.$$

Hence, by Lemma 3.5,

$$\frac{E_n^k(f)}{E_n(f)} \geqslant -1 - \frac{S_k N_k E_{n-k}(f^{(k)})}{n^k E_n(f)} + \frac{N_k \mid a_k \mid}{n^k E_n(f)}.$$

But $\lim_{n\to\infty} E_{n-k}(f^{(k)}) = 0$ and $a_k \neq 0$. Therefore

$$\frac{E_n^k(f)}{E_n(f)} \geqslant \frac{R_k}{n^k E_n(f)} - 1, \qquad n \geqslant 1. \tag{3.2}$$

Since $f \in C^k[a, b]$, Jackson's theorem implies

$$E_n(f) = o\left(\frac{1}{n^k}\right)$$

so that we have

$$\lim_{n\to\infty} \frac{E_n^k(f)}{E_n(f)} = \infty.$$

(3.2) and the relation

$$E_n(f) \leqslant \frac{K}{n} E_{n-1}(f')$$

complete the proof.

LEMMA 3.7. *Let* $f \in C^N[a, b]$, $N \geqslant 0$, $a < 0 < b$. *Suppose that* $|f^{(N)}(x) - f^{(N)}(y)| \leqslant K|x - y|^\epsilon$, $0 < \epsilon \leqslant 1$, $x, y \in [a, b]$. *Let* k *be an integer* $\geqslant N + 1$. *There exists a constant* K_k *satisfying*

$$E_n^k(f) \leqslant K_k \frac{1}{n^{N+\epsilon}}, \qquad n = 1, 2, \ldots .$$

Proof. Let a_n^k be as in Lemma 3.5. We have, as in the proof of Lemma 3.6,

$$E_n^k(f) \leqslant E_n(f) + |a_n^k| E_n^k(x^k).$$

Theorem 3.4 in [3] implies that

$$|a_n^k| \leqslant M_k n^{k-N+\epsilon}.$$

Thus

$$\frac{E_n^k(f)}{n^{-N-\epsilon}} \leqslant \frac{E_n(f)}{n^{-N-\epsilon}} + M_k N_k \frac{n^{k-N-\epsilon}}{n^k n^{-N-\epsilon}}$$

because, by Theorem 2.7,

$$E_n^k(x^k) \leqslant \frac{N_k}{n^k}.$$

But by Jackson's theorem, $E_n(f)/n^{-N-\epsilon}$ is bounded. The lemma follows.

Proof of Theorem 3.4. Let $g(x) = x^N |x|$. We have

$$E_n(g) \geqslant \frac{K_1}{n^{N+1}}.$$

On the other hand, by the preceding lemma,

$$E_n^k(g) \leqslant \frac{M_k}{n^{N+1}}.$$

IV. A Remark and a Conjecture

The relation

$$E_n^k(f) \leqslant E_n(f) + |a_n^k| E_n^k(x^k),$$

where a_n^k is as in Lemma 3.5, and the remark following the proof of Theorem 2.7 show that

$$\overline{\lim_{n\to\infty}} \frac{E_n^k(f)}{E_n(f)} < \infty$$

if $f \in C[a, b]$, $0 \notin [a, b]$, $E_n(f) \geqslant 1/C^{n^\alpha}$ for all $n \geqslant 1$, $C > 1$ and $\alpha < 1$.

We make the following conjecture: if $f \in C[-1, 1]$ and if f' does not exist at some interior point of $[-1, 1]$, then

$$\overline{\lim_{n\to\infty}} \frac{E_n^k(f)}{E_n(f)} < \infty.$$

Acknowledgments

I am grateful to Professors G. G. Lorentz and O. Shisha for their interest in my work.

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